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MAPPING PROPERTIES OF PLANAR HARMONIC FUNCTIONS VIA MILLER-ROSS FUNCTIONS

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Abstract: In this paper, we investigate the relationship between a novel class of harmonic univalent functions and the harmonic starlike and convex functions defined in the open unit disk, using a convolution operator associated with the Miller-Ross function. The importance of this investigation lies in the fact that convolution operators generated by special functions, such as the Miller–Ross function, provide powerful tools for constructing and characterizing new subclasses of harmonic mappings with rich geometric behavior. Several corollaries and related consequences of the main results are also established.

Keywords and Phrases: Analytic functions, Univalent functions, Miller-Ross function, harmonic functions.

2020 Mathematics Subject Classification: 30C45.

1. Introduction

Harmonic functions hold a prominent place in mathematical analysis and applied mathematics because of its exquisite qualities and wide-ranging applicability in many fields. Numerous mathematical phenomena can be discovered through the study of harmonic functions. Because both their real and imaginary parts are harmonic, holomorphic functions and harmonic functions are intimately related in complex analysis. Additionally, Harmonic functions are essential in conformal mappings, which allow geometries to be changed while maintaining angles—a critical function in domains such as image processing and aerodynamics. In addition to their theoretical importance, harmonic functions are used in many other contexts. They are employed in engineering to optimize signal processing and model electrical fields. In physics, they characterize states of equilibrium in systems that are subject to Laplace's equation.

A function $f(z) = u(z) + iv(z)$ that is twice continuously differentiable and fulfills $\Delta f(z) = 4f_{z\bar{z}}(z) = 0$ in D is a *complex-valued harmonic function* in a domain $D \subset \mathbb{C}$. The *canonical representation* of the function $f(z)$ is

$$f(z) = h(z) + \overline{g(z)}$$

when D is simply connected, where $h(z)$ and $g(z)$ are analytic functions in D and are termed the *analytic* and *co-analytic parts* of $f(z)$, respectively [11], [21], [23]. $f(z)$ is known to be *locally univalent* in D if and only if, the Jacobian

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 \neq 0,$$

and $f(z)$ is *sense-preserving* if $J_f(z) > 0$ in D. Consider $\mathcal{H}$ as the family of complex-valued harmonic functions $f(z)$ defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by the conditions

$$f(0) = 0,$$

and

$$f_z(0) = 1.$$

Here, the power series representations of $h(z)$ and $g(z)$ are as

$$h(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad g(z) = \sum_{j=1}^{\infty} b_j z^j, \quad z \in \mathbb{D}.$$

Thus, the function f can be written explicitly as

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j + \overline{\sum_{j=1}^{\infty} b_j z^j}, \quad z \in \mathbb{D}. \tag{1.1}$$

Define $S_{\mathcal{H}}$ as the family of all harmonic functions in $\mathcal{H}$ that are univalent and sense-preserving on $\mathbb{D}$. A related subclass, denoted by $S^{\circ}_{\mathcal{H}}$, is defined as follows:

$$S^{\circ}_{\mathcal{H}} = \{f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}} : g(0) = b_1 = 0\}.$$

The classes $S_{\mathcal{H}}^{\circ}$ and $S_{\mathcal{H}}$ are first studied in [12]. We also refer to the subclasses $S_{\mathcal{H}}^{*,\circ}$ and $K_{\mathcal{H}}^{\circ}$, which denote the subclasses of $S_{\mathcal{H}}^{\circ}$ of harmonic functions that are, respectively, starlike and convex in $\mathbb{D}$ [1, 2, 3, 10, 13].

2. Preliminary Results

A unique function called the *Miller–Ross function* was presented by Miller and Ross [25] as a framework for solving fractional-order initial value problem. The definition of this function is:

$$E_{v,c}(z) = z^c e^z Y^*(c, vz), \quad (c, v, z \in \mathbb{C},\ \Re(c) > 0,\ \Re(v) > 0),$$

where Y^* represents the incomplete gamma function ([25], p. 314). Utilizing the characteristics of the incomplete gamma function, the Miller–Ross function can be alternatively represented as:

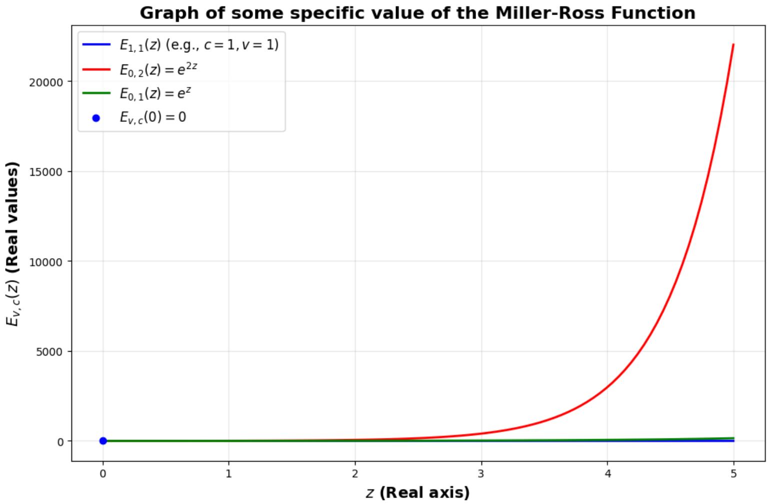
$$E_{v,c}(z) = z^c \sum_{j=0}^{\infty} \frac{(vz)^j}{\Gamma(j+c+1)}, \quad (c, v, z \in \mathbb{C},\ \Re(c) > 0,\ \Re(v) > 0). \tag{2.1}$$

The following are some examples of Miller–Ross function values:

$$\begin{aligned} &E_{v,c}(0) = 0, \quad (\Re(c) > 0), \\ &E_{0,v}(z) = e^{vz}, \\ &E_{0,1}(z) = e^z. \end{aligned}$$

The normalized Miller–Ross function $E_{v,c}$ is both univalent and starlike within the open disk $\mathbb{D}_{\frac{1}{2}} = \{z \in \mathbb{C} : |z| < \frac{1}{2}\}$ for $v > 0$ and $c > 2v - 1$, as recently demonstrated by Eker and Ece [14]. Furthermore, it was demonstrated that the normalized Miller–Ross function $E_{v,c}$ stays univalent and convex within the domain $\mathbb{D}_{\frac{1}{2}}$ if $c > (2 + \sqrt{2})v - 1$. Readers can refer to Miller and Ross's original work [25] for further information.

The Miller-Ross function is frequently used in the analysis of fractional-order differential and integral problems. It is a vital tool in many domains, including the fractional representation of kinetic equations, modeling random walks, Lévy flights, super-diffusive transport processes, and the analysis of complex systems. Numerous studies, such as those in [4, 6, 8, 19, 22, 26, 28, 29, 38], examine its properties, applications, and generalizations.



The Miller Rose function $E_{v,c}(z) \notin \mathcal{A}$ is clearly visible. Next, in accordance with Eker and Ece [14], we have the following normalization:

$$\begin{aligned}\mathbb{E}_{v,c}(z) &= z^{1-v}\Gamma(v+1)E_{v,c}(z)\\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} z^j,\end{aligned} \tag{2.2}$$

where $z, c \in \mathbb{C}$ with $v > -1$.

We will now refer to the $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$ class established by Porwal et al. [35]. For $0 \leq \lambda \leq 1$, $0 \leq \varepsilon < 1$, and $\alpha \in \mathbb{R}$

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{(1-\lambda)z' + \lambda f(z)} - e^{i\alpha}\right) > \varepsilon, \tag{2.3}$$

where $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$, and $f'(z) = \frac{\partial}{\partial\theta} f(re^{i\theta})$.

The inequality (2.3) can be written as:

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))' - \overline{z(g(z))'}}{(1-\lambda)z + \lambda\left[h(z) + \overline{g(z)}\right]} - e^{i\alpha}\right) > \varepsilon, \quad z \in \mathbb{D}. \tag{2.4}$$

Example 2.1. [35] For $\lambda = 0$ and $0 \leq \varepsilon < 1$, we obtain $W_{\mathcal{H}}(0, \alpha, \varepsilon) \equiv N_{\mathcal{H}}(\alpha, \varepsilon)$, meeting the analytical requirements

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{z'} - e^{i\alpha}\right) > \varepsilon, \quad \alpha \in \mathbb{R},$$

equivalently, we obtain

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))'-\overline{z(g(z))'}}{z}-e^{i\alpha}\right)>\varepsilon,\quad z\in\mathbb{D}.$$

Example 2.2. [36] For $\lambda=1$, $0\leq\varepsilon<1$, we have $W_{\mathcal{H}}(1,\alpha,\varepsilon)\equiv R_{\mathcal{H}}(\alpha,\varepsilon)$, meeting the analytical requirements

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{f(z)}-e^{i\alpha}\right)>\varepsilon,\quad \alpha\in\mathbb{R},$$

equivalently, this can be written as

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))'-\overline{(z(g(z))')}}{h(z)+\overline{g(z)}}-e^{i\alpha}\right)>\varepsilon,\quad z\in\mathbb{D}.$$

Also, let

$$\begin{aligned}WT_{\mathcal{H}}(\lambda,\alpha,\varepsilon)&=W_{\mathcal{H}}(\lambda,\alpha,\varepsilon)\cap T_{\mathcal{H}},\\ RT_{\mathcal{H}}(\alpha,\varepsilon)&=R_{\mathcal{H}}(\alpha,\varepsilon)\cap T_{\mathcal{H}},\\ NT_{\mathcal{H}}(\alpha,\varepsilon)&=N_{\mathcal{H}}(\alpha,\varepsilon)\cap T_{\mathcal{H}},\end{aligned}$$

where $T_{\mathcal{H}}$ is the subclass of $\mathcal{H}$ consisting of harmonic functions $f=h+\bar{g}$ of the form:

$$f(z)=z-\sum_{j=2}^{\infty}|a_j|\,z^j+\sum_{j=1}^{\infty}|b_j|\,\overline{z^j}.$$

To find our main results, the following lemma are required.

Lemma 2.1. [35] *For $a_1=1$, $0\leq\varepsilon<1$, $f=h+\bar{g}$ belongs to $W_{\mathcal{H}}(\lambda,\alpha,\varepsilon)$ if and only if*

$$\sum_{j=2}^{\infty}\frac{2j-\lambda(1+\varepsilon)}{1-\varepsilon}|a_j|+\sum_{j=1}^{\infty}\frac{2j+\lambda(1+\varepsilon)}{1-\varepsilon}|b_j|\leq 1.$$

Lemma 2.2. [35] *For $a_1=1$, $0\leq\varepsilon<1$, $f\in NT_{\mathcal{H}}(\alpha,\varepsilon)$ iff*

$$\sum_{j=2}^{\infty}\frac{2j}{1-\varepsilon}|a_j|+\sum_{j=1}^{\infty}\frac{2j}{1-\varepsilon}|b_j|\leq 1.$$

Lemma 2.3. [36] *For $a_1=1$, $0\leq\varepsilon<1$, $f=h+\bar{g}\in RT_{\mathcal{H}}(\alpha,\varepsilon)$ if and only if:*

$$\sum_{j=2}^{\infty}\frac{2j-1-\varepsilon}{1-\varepsilon}|a_j|+\sum_{j=1}^{\infty}\frac{2j+1+\varepsilon}{1-\varepsilon}|b_j|\leq 1.$$

Lemma 2.4. [12, 13] *If $f = h + \bar{g} \in K^{\circ}_{\mathcal{H}}$ with $b_1 = 0$, then*

$$|a_j| \le \frac{j+1}{2}, \quad |b_j| \le \frac{j-1}{2}, \quad j \ge 2.$$

Lemma 2.5. [12, 13] *If $f = h + \bar{g} \in S^{*,\circ}_{\mathcal{H}}$ with $b_1 = 0$, then*

$$|a_j| \le \frac{(2j+1)(j+1)}{6}, \quad |b_j| \le \frac{(2j-1)(j-1)}{6}.$$

Further, we recall the following remarks.

Remark 2.1. *In* [35], *it is also shown that $f \in W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$ iff*

$$|a_j| \le \frac{1-\varepsilon}{2j - \lambda(1+\varepsilon)}, \quad j \ge 2,$$

$$|b_j| \le \frac{1-\varepsilon}{2j + \lambda(1+\varepsilon)}, \quad j \ge 1.$$

Putting $\lambda = 0$, it is expressed as follows

Remark 2.2. *Assume $0 \le \varepsilon < 1$ and $\alpha \in \mathbb{R}$. Then, the function $f \in NT_{\mathcal{H}}(\alpha, \varepsilon)$ if and only if the following criteria are satisfied*

$$|a_j| \le \frac{1-\varepsilon}{2j} \quad \text{and} \quad |b_j| \le \frac{1-\varepsilon}{2j}, \quad j \ge 2.$$

Taking $\lambda = 1$, we have

Remark 2.3. [36] *Let $0 \le \varepsilon < 1$ and $\alpha \in \mathbb{R}$. Then $f \in RT_{\mathcal{H}}(\alpha, \varepsilon)$ if and only if*

$$|a_j| \le \frac{1-\varepsilon}{2j-1-\varepsilon} \quad \text{and} \quad |b_j| \le \frac{1-\varepsilon}{2j+1+\varepsilon}, \quad j \ge 2.$$

We will continue to focus on the situation of real-valued v, c, and $z \in \mathbb{C}$ in this study. Consider the functions

$$\mathbb{E}_{v,c}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} z^j, \quad \mathbb{E}_{u,k}(z) = z + \sum_{j=1}^{\infty} \frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)} z^j.$$

The following describes the way $\mathfrak{F}(f)$ is defined by Murugurusundaramoorthy et al. [32]:

$$\mathcal{F}(z) = \mathfrak{F}f(z) = h(z) * \mathbb{E}_{v,c}(z) + \overline{g(z) * \mathbb{E}_{u,k}(z)} \tag{2.5}$$

$$\mathcal{F}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} a_j z^j + \overline{\sum_{j=1}^{\infty} \frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)} b_j z^j}, \tag{2.6}$$

for real parameters v, c, u, k with $v, c, u, k \notin \{0, -1, -2, \dots\}$.

Using hypergeometric functions [20, 21] and recent studies pertaining to distribution series [9, 16, 17, 18, 29, 33, 37], inclusion relations between various subclasses of analytic and univalent functions were examined in the literature. For the families of harmonic univalent functions, including different linear and nonlinear operators, a number of authors have recently looked into mapping properties and inclusion results [5, 7, 15, 24, 27, 30, 31, 34, 35, 39, 40]. Inspired by the previously mentioned works, we used the convolution operator associated with the Miller-Ross function in our present paper to determine the connections between the classes $S_{\mathcal{H}}^{*0}$, $K_{\mathcal{H}}^{0}$, and $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$.

3. Main Results

We contribute to the current studies in harmonic univalent function theory by extending the results and establishing connections among the classes $S_{\mathcal{H}}^{*0}$, $K_{\mathcal{H}}^{0}$, and $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$ using the convolution operator $\mathcal{F}$ defined via the Miller–Ross function. In this research, we define:

$$\mathbb{E}_{v,c}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} z^j,$$

and for $z = 1$

$$\mathbb{E}_{v,c}(1) = 1 + \sum_{j=2}^{\infty} \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}, \tag{3.1}$$

$$\mathbb{E}'_{v,c}(1) = 1 + \sum_{j=2}^{\infty} \frac{j\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}, \tag{3.2}$$

$$\mathbb{E}''_{v,c}(1) = \sum_{j=2}^{\infty} \frac{j(j-1)\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}, \tag{3.3}$$

and

$$\mathbb{E}'''_{v,c}(1) = \sum_{j=2}^{\infty} \frac{j(j-1)(j-2)\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}. \tag{3.4}$$

The inclusion relations among the harmonic classes $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$, $S_{\mathcal{H}}^{*0}$, and $K_{\mathcal{H}}^{0}$ are established in this section using the Miller-Ross function.

Theorem 3.1. *Assume* $0 \le \varepsilon < 1$, *and* $v, c, u, k \notin \{0, -1, -2, \dots\}$ *be real numbers. If the inequality*

$$\begin{aligned} &2E''_{v,c}(1) + (4 - \lambda(1+\varepsilon))E'_{v,c}(1) - \lambda(1+\varepsilon)E_{v,c}(1) \\ &+ 2E''_{u,k}(1) + \lambda(1+\varepsilon)E'_{u,k}(1) - \lambda(1+\varepsilon)E_{u,k}(1) \\ &\le 2[(1-\lambda)(1+\varepsilon) + 2(1-\varepsilon)] \end{aligned}$$

is true, then we have the inclusion $F(K^{\circ}_{\mathcal{H}}) \subset W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$.

Proof. Assuming $b_1 = 0$, consider $f = h + \overline{g} \in K^{\circ}_{\mathcal{H}}$. As stated in (2.6), where $b_1 = 0$, our goal is to demonstrate that $F(z) = F(f)$ belongs to $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$. Therefore, Lemma 2.1 is sufficient to illustrate the following inequality:

$$\begin{aligned} f_1 = &\sum_{j=2}^{\infty} [2j - \lambda(1+\varepsilon)] \left| \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} a_j \right| \\ &+ \sum_{j=2}^{\infty} [2j + \lambda(1+\varepsilon)] \left| \frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)} b_j \right| \le 1 - \varepsilon. \end{aligned}$$

Using Lemma 2.4, we now establish the relationship between harmonic convex functions and $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$:

$$f_1 \le \frac{1}{2} \left\{ \sum_{j=2}^{\infty} (2j - \lambda(1+\varepsilon))(j+1) \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} + \sum_{j=2}^{\infty} (2j + \lambda(1+\varepsilon))(j-1) \frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)} \right\}.$$

Simplifying further, we have

$$\begin{aligned} f_1 = &\sum_{j=2}^{\infty} \left[j - \frac{\lambda}{2}(1+\varepsilon) \right] (j+1) \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} \\ &+ \sum_{j=2}^{\infty} \left[j + \frac{\lambda}{2}(1+\varepsilon) \right] (j-1) \frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}. \end{aligned}$$

After expressing $j^2 = j(j-1) + j$ and $j = (j-1) + 1$, we arrive

$$\begin{aligned} f_1 = &\sum_{j=2}^{\infty} \left[(j-1)(j-2) + \left(4 - \frac{\lambda(1+\varepsilon)}{2}\right)(j-1) \right] \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} \\ &+ \sum_{j=2}^{\infty} 2\left(1 - \frac{\lambda(1+\varepsilon)}{2}\right) \frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} \\ &+ \sum_{j=2}^{\infty} \left[(j-1)(j-2) + \left(2 + \frac{\lambda(1+\varepsilon)}{2}\right)(j-1) \right] \frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}. \end{aligned}$$

Now, using (3.1), (3.2) and (3.3), we have

$$\begin{aligned} f_1 \leq{}& E''_{v,c}(1) - 2E'_{v,c}(1) + 2E_{v,c}(1) + \left(4 - \frac{\lambda}{2}(1+\varepsilon)\right)\left[E'_{v,c}(1) - E_{v,c}(1)\right] \\ &+ 2\left(1 - \frac{\lambda}{2}(1+\varepsilon)\right)\left[E_{v,c}(1) - 1\right] + E''_{u,k}(1) - 2E'_{u,k}(1) + 2E_{u,k}(1) \\ &+ \left(2 + \frac{\lambda}{2}(1+\varepsilon)\right)\left[E'_{u,k}(1) - E_{u,k}(1)\right]. \end{aligned}$$

An easy computation shows that

$$\begin{aligned} f_1 ={}& E''_{v,c}(1) + \left(2 - \frac{\lambda}{2}(1+\varepsilon)\right)E'_{v,c}(1) - \frac{\lambda}{2}(1+\varepsilon)E_{v,c}(1) \\ &+ E''_{u,k}(1) + \frac{\lambda}{2}(1+\varepsilon)E'_{u,k}(1) - \frac{\lambda}{2}(1+\varepsilon)E_{u,k}(1) \\ \leq{}& (1-\lambda)(1+\varepsilon) + 2(1-\varepsilon). \end{aligned}$$

The final expression does not exceed $1-\varepsilon$ when the stated condition holds. Hence, the proof of Theorem 3.1 is concluded.

Remark 3.1. *In Theorem 3.1, setting $\lambda = 1$ enhances (by setting $G = 1$) the outcome of Theorem 2.1 in* [32].

Building upon Theorem 3.1, we derive the necessary condition for the class $S^{*,\circ}_{\mathcal{H}}$ in relation to $W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$.

Theorem 3.2. *Let $0 \leq \varepsilon < 1$, and v, c, u, k be real values such that $v, c, u, k \notin \{0, -1, -2, \dots\}$. If the inequality*

$$\begin{aligned} &4E'''_{v,c}(1) + 2[9 - \lambda(1+\varepsilon)]E''_{v,c}(1) + [12 - 5\lambda(1+\varepsilon)]E'_{v,c}(1) - \lambda(1+\varepsilon)E_{v,c}(1) \\ &+ 4E'''_{u,k}(1) + 2[3 + \lambda(1+\varepsilon)]E''_{u,k}(1) - \lambda(1+\varepsilon)E'_{u,k}(1) + \lambda(1+\varepsilon)E_{u,k}(1) \\ &\leq 3[4 - (1+\lambda)(1+\varepsilon)]. \end{aligned}$$

is satisfied, then $\mathcal{S}^{*,\circ}_{\mathcal{H}} \subset \mathcal{W}_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$.

Proof. Assume $f = h + \overline{g} \in \mathcal{S}^{*,\circ}_{\mathcal{H}}$ with $b_1 = 0$. We aim to prove that $F(z) = F(f) \in W_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$, as described by (2.6) under the condition $b_1 = 0$. Using Lemma 2.1, it is sufficient to show that:

$$\begin{aligned} f_2 ={}& \sum_{j=2}^{\infty} [2j - \lambda(1+\varepsilon)]\left|\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}a_j\right| \\ &+ \sum_{j=2}^{\infty} [2j + \lambda(1+\varepsilon)]\left|\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}b_j\right| \leq 1 - \varepsilon. \end{aligned}$$

Using Lemma 2.5, we have

$$f_2 \leq \frac{1}{3}\left\{\sum_{j=2}^{\infty}\left[\left(j-\frac{\lambda}{2}\right)(1+\varepsilon)\right](2j+1)(j+1)\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}\right.$$
$$\left.+\sum_{j=2}^{\infty}\left[\left(j+\frac{\lambda}{2}(1+\varepsilon)\right)\right](2j-1)(j-1)\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}\right\}.$$

Next, expressing $j^3 = (j-1)(j-2)(j-3) + 6(j-1)(j-2) + 7(j-1) + 1$, $j^2 = j(j-1) + j$ and $j = (j-1) + 1$, we arrive

$$f_2 = \frac{1}{3}\left\{2\sum_{j=2}^{\infty}(j-1)(j-2)(j-3)\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}\right.$$
$$+[15-\lambda(1+\varepsilon)]\sum_{j=2}^{\infty}(j-1)(j-2)\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}$$
$$+\left[24-\frac{9\lambda}{2}(1+\varepsilon)\right]\sum_{j=2}^{\infty}(j-1)\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}$$
$$+6\left[1-\frac{\lambda}{2}(1+\varepsilon)\right]\sum_{j=2}^{\infty}\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}$$
$$+2\sum_{j=2}^{\infty}(j-1)(j-2)(j-3)\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}$$
$$+[9+\lambda(1+\varepsilon)]\sum_{j=2}^{\infty}(j-1)(j-2)\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}$$
$$\left.+\left[6+\frac{3\lambda}{2}(1+\varepsilon)\right]\sum_{j=2}^{\infty}(j-1)\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}\right\}.$$

Using (3.1), (3.2), (3.3) and (3.4), we find

$$f_2 \leq \frac{1}{3}\left\{2\left[E'''_{v,c}(1) - 3E''_{v,c}(1) + 6E'_{v,c}(1) - 6E_{v,c}(1)\right]\right.$$
$$+[15-\lambda(1+\varepsilon)]\left[E''_{v,c}(1) - 2E'_{v,c}(1) + 2E_{v,c}(1)\right]$$
$$+\left[24-\frac{9\lambda}{2}(1+\varepsilon)\right]\left[E'_{v,c}(1) - E_{v,c}(1)\right] + 6\left[1-\frac{\lambda}{2}(1+\varepsilon)\right]\left[E_{v,c}(1) - 1\right]$$

$$+ 2\left[E'''_{u,k}(1) - 3E''_{u,k}(1) + 6E'_{u,k}(1) - 6E_{u,k}(1)\right]$$
$$+ [9+\lambda(1+\varepsilon)]\left[E''_{u,k}(1) - 2E'_{u,k}(1) + 2E_{u,k}(1)\right]$$
$$+ \left[6 + \frac{3\lambda}{2}(1+\varepsilon)\right]\left[E'_{u,k}(1) - E_{u,k}(1)\right]\Bigg\},$$

or, equivalently

$$f_2 = \frac{1}{3}\Bigg\{2E'''_{v,c}(1) + [9-\lambda(1+\varepsilon)]\,E''_{v,c}(1) + \left[6 - \frac{5\lambda}{2}(1+\varepsilon)\right]E'_{v,c}(1)$$
$$- \frac{\lambda}{2}(1+\varepsilon)E_{v,c}(1) + \left[-6+3(1+\varepsilon)\right] + 2E'''_{u,k}(1)$$
$$+ \left[3+\lambda(1+\varepsilon)\right]E''_{u,k}(1) - \frac{\lambda}{2}(1+\varepsilon)E'_{u,k}(1) + \frac{\lambda}{2}(1+\varepsilon)E_{u,k}(1)\Bigg\} \le 1-\varepsilon.$$

This completes the proof.

Remark 3.2. *Setting $\lambda = 1$ in Theorem 3.2 improves (by setting $G = 1$) the result obtained in Theorem 2.2 in* [32].

Theorem 3.3. *Assume $0 \le \varepsilon < 1$ and v, c, u, k are real values such that $v, c, u, k \notin \{0, -1, -2, \dots\}$. If the inequality*

$$E_{v,c}(1) + E_{u,k}(1) \le 2$$

is true, then $F(W_{\mathcal{H}}(\lambda,\alpha,\varepsilon)) \subset W_{\mathcal{H}}(\lambda,\alpha,\varepsilon)$.
Proof. Consider $f = h + \overline{g} \in W^{\circ}_{\mathcal{H}}(\lambda,\alpha,\varepsilon)$. Utilizing Lemma 2.1, it is sufficient to establish that

$$f_3 = \sum_{j=2}^{\infty}[2j-\lambda(1+\varepsilon)]\left|\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)}a_j\right| + \sum_{j=2}^{\infty}[2j+\lambda(1+\varepsilon)]\left|\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}b_j\right| \le 1-\varepsilon.$$

By Remark 3.1, we have

$$f_3 \le (1-\varepsilon)\left\{\sum_{j=2}^{\infty}\frac{\Gamma(v+1)c^{j-1}}{\Gamma(v+j)} + \sum_{j=1}^{\infty}\frac{\Gamma(u+1)k^{j-1}}{\Gamma(u+j)}\right\}.$$

By using equation (3.1), finally we get

$$f_3 \le (1-\varepsilon)\left[E_{v,c}(1) - 1 + E_{u,k}(1)\right],$$

which satisfies the given condition.

Setting $\lambda = 0$ in Theorems 3.1 and 3.2 yields the following results:

Corollary 3.1. *Let $0 \leq \varepsilon < 1$ and v, c, u, k be real values such that $v, c, u, k \notin \{0, -1, -2, \dots\}$. If the inequality*

$$E''_{v,c}(1) + 2E'_{v,c}(1) + E''_{u,k}(1) \leq 3 - \varepsilon$$

is true, then $F(K^{\circ}_{\mathcal{H}}) \subset N_{\mathcal{H}}(\alpha, \varepsilon)$.

Corollary 3.2. *Let $0 \leq \varepsilon < 1$ and v, c, u, k be real values such that $v, c, u, k \notin \{0, -1, -2, \dots\}$. If the inequality*

$$4E'''_{v,c}(1) + 18E''_{v,c}(1) + 12E'_{v,c}(1) + 4E'''_{u,k}(1) + 6E''_{u,k}(1) \leq 3(3 - \varepsilon)$$

is true, then $F(S^{*,\circ}_{\mathcal{H}}) \subset N_{\mathcal{H}}(\alpha, \varepsilon)$.

4. Conclusion

The main objective of the paper is to find some inclusion relations of the harmonic classes $\mathcal{W}_{\mathcal{H}}(\lambda, \alpha, \varepsilon)$ with subclasses $\mathcal{S}^{\circ}_{\mathcal{H}}$ of harmonic functions $\mathcal{S}^{*,\circ}_{\mathcal{H}}$ and $\mathcal{K}^{\circ}_{\mathcal{H}}$, which are starlike and convex, respectively. Utilizing the operator $\mathcal{F}$, researchers might be inspired to find new inclusion relations for new harmonic classes of analytic functions with the classes $\mathcal{S}^{*,\circ}_{\mathcal{H}}$ and $\mathcal{K}^{\circ}_{\mathcal{H}}$. These findings have applications in geometric function theory, conformal and harmonic mapping problems, image processing, and physical models involving potential flows, where identifying mappings with controlled geometric properties is essential.

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